**B.A./B.Sc. 3rd Semester (Honours) Examination, 2022 (CBCS)**

**Subject : Mathematics**

**Course : CC-V (BMH3CC05)**

**(Theory of Real Function & Introduction to Metric Space)**

**Time: 3 Hours** **Full Marks: 60**

*The figures in the margin indicate full marks.*
*Candidates are required to give their answers in their own words as far as practicable.*
*(Notations and symbols have their usual meaning.)*

1. Answer *any ten* questions: 2×10=20

(a) Prove that if $a > 0$, then $\lim_{x\to 0^+} \frac{x}{a}\left[\frac{b}{x}\right] = \frac{b}{a}$ , where $[x]$ is the greatest integer in $x$ but not greater than $x$.

(b) Let $f: [a, b] \to [a, b]$ be continuous. Prove that there exists $c$ in $[a, b]$ such that $f(c) = c$ .

(c) Give an example to show that a continuous function on a closed interval $I$ may not be bounded on $I$.

(d) Let $f$ be continuous in $[a, b]$ and let $f(c) < 0$ where $a < c < b$ . Show that there exists a sub-interval $(c - \delta, c + \delta)$ of $[a, b]$ in which every $f(x)$ is negative.

(e) Give the geometrical significance of Rolle's theorem.

(f) Show that $a^x > x^a$ if $x > a \geq e$.

(g) If $f'$ exists and is bounded on some interval $I$, then prove that $f$ is uniformly continuous on $I$.

(h) Give an example of a function defined on $\mathbb{R}$ such that $f$ is not differentiable at $1,2, \ldots, 100$ only.

(i) If $f: \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and $f(x) = 0$ for all $x \in Q$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

(j) Let $f(x + y) = f(x)f(y) \forall x, y \in \mathbb{R}$ and $f'(0)$ exists and $f'(0) \neq 0$. Show that $f$ is differentiable at every point $x \in \mathbb{R}$ and $f'(x) = f(x)f'(0)$.

(k) Describe an open ball of unit radius about $\frac{1}{2}$ for the metric $d$, defined by $d(x, y) = |x - y| \ \forall \ x, y \in [0,3]$.

(l) Let $v = \{(x, y) \in \mathbb{R}^2 : x \notin \mathbb{Z},\ y \notin \mathbb{Z}\}$. Is $U$ an open set in $\mathbb{R}^2$, with respect to the usual metric? Justify.

(m) Let $A = \{(x, y) : x^2 + y^2 = 1\}$ and $B = \{x, y) : (x - 1)^2 + y^2 = 1\}$. Find the diameters of the sets $A \cup B$ and $A \cap B$. 1+1

(n) A function $\rho$ is defined by $\rho(x,y) = |x^2 - y^2|$. Examine if $\rho$ is a metric on $(-\infty,\infty)$.

(o) Give an example to show that arbitrary intersection of open sets need not be open.

2. Answer *any four* questions: 5×4=20

(a) State and prove Darboux's theorem on derivative. 1+4=5

(b) (i) A function $f: \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x, & \text{if } x \in Q \\ 0, & \text{if } x \in \mathbb{R} - Q \end{cases}$$

Show that $f$ is discontinuous at every non-zero real number.

(ii) Evaluate $\lim_{x\to 0} x\left[\frac{1}{x}\right]$, if exists, where $[m]$ is the largest integer less than or equal to $m$.

3+2=5

(c) (i) Let $f$ be continuous function on a closed bounded interval $[a,b]$. Show that $f$ assumes every value between $m$ and $M$, where $m = \text{Inf}\{f(x) : x \in [a,b]\}$ and $M = \text{Sup}\{f(x) : x \in [a,b]\}$.

(ii) Prove that $\frac{2x}{\pi} < \sin x$ for $x \in \left(0, \frac{\pi}{2}\right)$.

3+2=5

(d) (i) A function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(0) = 0$ and

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is an irrational number} \\ \frac{1}{q}, & \text{if } x = \frac{p}{q}, \text{where } p \in \mathbb{Z}, q \in \mathbb{N} \text{ and } \gcd(p,q) = 1. \end{cases}$$

Show that $f$ is not differentiable at $x = 0$.

(ii) A function $f: \mathbb{R} \to \mathbb{R}$ satisfies the condition $|f(x) - f(y)| \le (x-y)^2, \forall\, x,y \in \mathbb{R}$.

Prove that $f$ is constant function on $\mathbb{R}$.

3+2=5

(e) (i) Find the point on the parabola $y^2 = 8x$ at which the radius of curvature is $\frac{125}{16}$.

(ii) Use Taylor's theorem to prove that

$$1 + \frac{x}{2} - \frac{x^3}{8} < \sqrt{1+x} < 1 + \frac{x}{2} \text{ if } x > 0.$$

3+2=5

(f) (i) If $(X,d)$ is a metric space, then show that $(X,d^*)$ is a metric space, where

$$d^*(x,y) = \frac{d(x,y)}{1+d(x,y)} \forall\, x,y \in X.$$

(ii) Let $(\mathbb{R},d)$ be the real number space with usual metric. Is $(\mathbb{R},d)$ separable? Give reason.

3+2=5

3. Answer *any two* questions: 10×2=20

(a) (i) The tangents at two points $P$ and $Q$ on the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ are at right angles. Show that if $\rho_1$ and $\rho_2$ be the radii of curvature at these points, then $\rho_1^2 + \rho_2^2 = 16a^2$.

(ii) Assuming that $f''(x)$ exists for all $x$ in $[a, b]$, show that

$f(c) - f(a)\frac{b-c}{b-a} - f(b)\frac{c-a}{b-a} - \frac{1}{2}(c-a)(c-b)f''(\xi) = 0$ where $c$ and $\xi$ both lie in $(a, b)$. 5+5=10

(b) (i) Let $I$ be an interval and a function $f: I \to \mathbb{R}$ be differentiable on $I$. Show that $f'(I)$ is an interval, $f'(I) = \{f'(x): x \in I\}$.

(ii) State and prove Rolle's theorem.

(iii) Use Mean Value Theorem to prove that

$0 < \frac{1}{x}\log\frac{e^x - 1}{x} < 1$ for $x > 0$. 3+(1+3)+3=10

(c) (i) Verify Maclaurin's infinite series expansion of the function

$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$ for $x \in (-1, 1]$.

(ii) If $f$ is continuous in $[a, b]$ then prove that $f$ is uniformly continuous in $[a, b]$.

(iii) A function $f$ is thrice differentiable on $[a, b]$ and $f(a) = f(b) = 0$,

$f'(a) = f'(b) = 0$. Prove that $f'''(c) = 0$ for some $c \in (a, b)$. 4+3+3=10

(d) (i) Let '$d$' be a metric on a set $X$. Prove that '$d_1$' is also a metric on $X$,

where $d_1(x, y) = \sqrt{d(x, y)}$, for all $x, y \in X$.

(ii) Prove that every closed sphere in a metric space is a closed set.

(iii) Let $X$ denote the set of all real sequences. For $x = \{x_n\} \in X$, $y = \{y_n\} \in X$ define

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{(2022)^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}.$$

Show that $d$ is a metric on $X$. 3+3+4=10